Statistical Thinking and Transnumeration

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This study explored transnumeration-type thinking exhibited by eleven and twelve year old students as they worked on a statistical investigation. Some aspects of this thinking are identified and described. The analysis raises issues about the development of such thinking and about how teaching might build on students' thinking strategies.

Statistical thinking is a complex intellectual activity that is "an artefact of civilisation, not part of our natural neural equipment" (Moore, 1998, p. 1257). This particular way of thinking and reasoning must be learnt. Wild and Pfannkuch (1999) in their framework for statistical thinking proposed five types of statistical thinking that they considered were fundamental in empirical enquiry. One of these was transnumeration, a coined word meaning "changing representations to engender understanding". If a modelling perspective is taken of statistics then we have a real world situation from which a statistical model of the situation is developed and transnumeration occurs in three instances: firstly, when quantitative or classification measures that are relevant to the problem are captured from the real world situation; secondly, in the statistical system when multiple representations of the data are employed in an attempt to understand what the data are saying about the real world situation; thirdly, when the statistical summaries are communicated in a form that is understandable to and convincing for the intended audience, and that is related to the original problem situation.

Many questions can be raised about how students learn and understand this type of thinking. Some of these are: What transnumeration-type thinking do students use? Are there transition or developmental stages in this thinking? How is a transnumeration structure of thought built up in teaching? Related research in this area on how students organise, reduce, represent, and interpret data has been conducted by many researchers such as Ben-Zvi and Friedlander (1997), Bright and Friel (1998), Cobb (1999), Moritz (2000), and Lehrer and Schauble (2000). Their research will be drawn upon to elucidate transnumeration-type thinking, a way of thinking that suggests how to change real and statistical data into other types of representations that may allow insights into the data and consequently the real situation.

Background of Study

This exploratory study was conducted by Rubick (2000) for her masters thesis. Her purpose was to ascertain whether all four dimensions of the Wild and Pfannkuch (1999) framework for statistical thinking in empirical enquiry could be used to characterise school students' thinking when they were conducting an investigation. The reason for such an exploration was that the framework had been created from statisticians describing their thinking when involved in projects, and she wanted to see if it were applicable to the

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students' statistical thinking at a macro-level. Yoon (2001), with the permission of Rubick, analysed three of the interviews to identify and describe at a micro-level the conceptions of the students using part of the second dimension of the framework, namely "types fundamental to statistical thinking" which included, transnumeration, consideration of variation, reasoning with statistical models, and integrating the statistical and the contextual. The analysis reported here takes three interviews, which all three authors analysed, and delineates some categories of transnumeration-type thinking that can be observed.

Method of Study

Twelve Year 7 and 8 students (11–13 year olds) were randomly selected from the most able students in a high socio-economic level school. The students were put into pairs so that members of the pair were of the same year level and gender, and were known to be able to work together. Using an investigation and protocol created by Watson, Collis, Callingham, and Moritz (1995), the students were given sixteen cards. Each card contained information about one person: the name, age, weight, eye colour, favourite activity and number of fast food meals eaten per week. The students had not worked with multivariate datasets before and were not familiar with scatterplots. The students were first required to read and understand the information on the cards before thinking about what they could investigate. A pencil-and-paper environment was provided with materials such as calculators and graph paper being available for the students. They were interviewed and audio-taped for approximately one hour while they were conducting an investigation of their choice. The students were asked to think aloud as they progressed through the investigation.

Analysis

The focus of this analysis is on transnumeration thinking not on how students engage with the other types of statistical thinking identified by Wild and Pfannkuch (1999). An analysis of the interview data revealed that it was possible to describe the students'

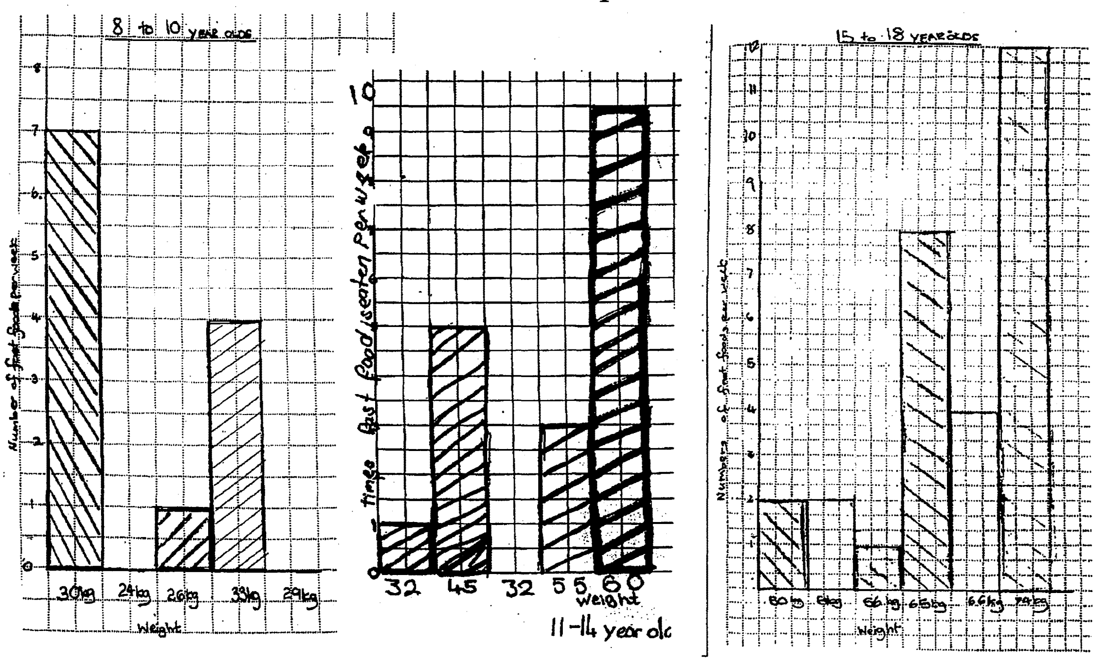

Figure 1. Graphs of Interview One students.

transnumeration thinking within the broad categories: classifying and reclassifying the data; calculating averages and constructing graphs; and communicating findings. To illustrate the range of thinking extracts will be drawn from three interviews. These will be referred to as Interview One students who grouped the data on age (Figure 1), Interview Two students who grouped the data on age and gender and found averages (Figure 2), and Interview Three students who grouped the data on age and activity and found averages and ratios (Figures 3, 4, & 5).

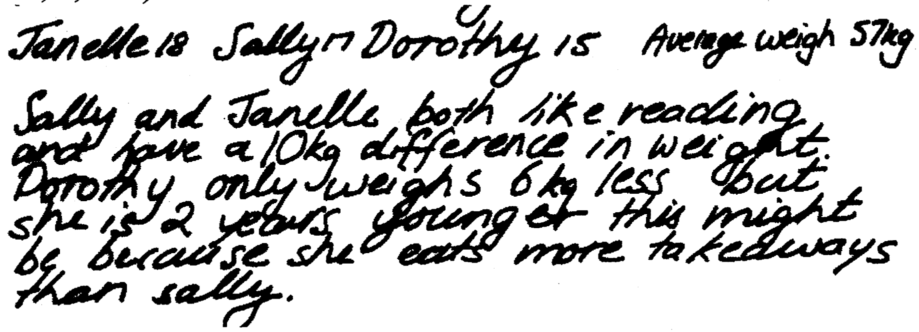

Figure 2. A part of Interview Two students' conclusion.

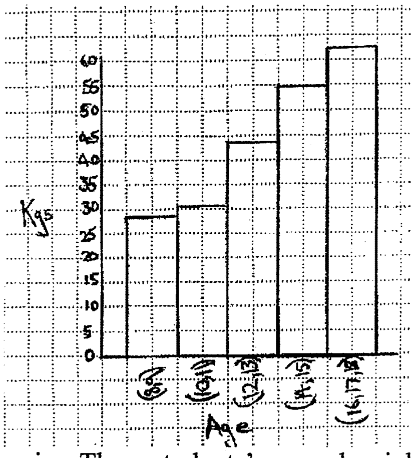

Figure 3. Interview Three students' age and weight graph.

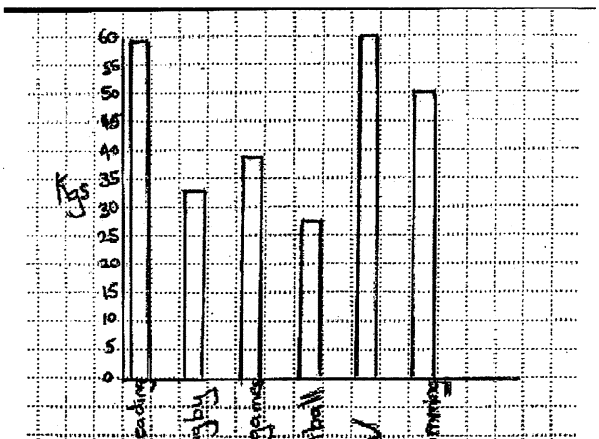

Figure 4. Interview Three students' activity and weight graph.

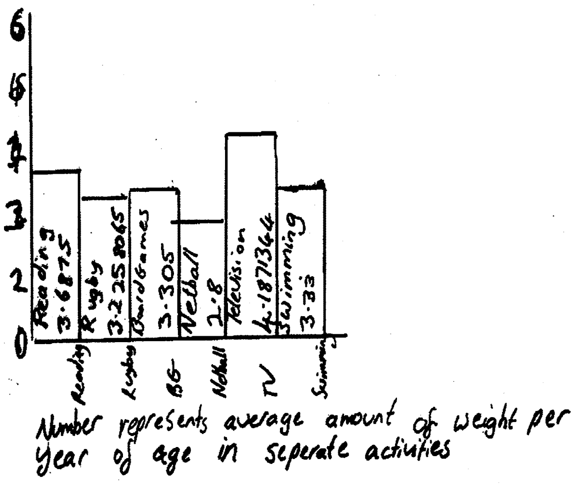

Figure 5. Interview Three students' activity and weight ratio graph.

After spending some time understanding the dataset these students took weight as the response variable and investigated possible explanatory variables. The concern was whether weight could be explained by the amount of fast food eaten per week (Interview 1), by fast food eaten or activity or age (Interview 2) or by activity or age (Interview 3).

Classifying and Reclassifying the Data

Transnumeration thinking was first prompted through the students sorting the cards on a particular characteristic such as age. The students in Interviews One and Three sorted the cards into groups of active and passive activities but were unable to reclassify the data and define a new variable. The Interview Two students sorted on gender, but did not define it as a variable. Only two students in one interview (not discussed here) defined gender as a variable. The students in these interviews (1, 2 and 3) sorted the data into distinct age groups but only Interview Three students were able to reclassify age group as a qualitative variable. The thinking that is required from sorting the data into groups based on a common characteristic to reclassifying data and defining new variables seems to be an identifiable transition step. The classification, reclassification, or stratification of data, involves not only determining new variables but also understanding how data can be structured into quantitative and qualitative variables. Recognising and representing implicit information in the data do not appear easy for the students. A tool that may help in the classification process is the construction of a table.

The transnumeration of the card data to a table representation seems to indicate there may be three levels of response. When considering all the students interviewed, the tables created by them could be classified as: (1) abbreviated form, very similar to how the original data were presented (two students); (2) some grouping of variables evident, data partially ordered (seven students – includes the students discussed here); (3) close to a conventional format, data ordered (two students). One student did not construct a table but instead established explanations and reasons for the observed phenomena. We believe that the transnumeration thinking that led to the construction of a table was a learnt procedure.

Calculating Averages and Constructing Graphs

Once the data are classified and organised transnumeration thinking is operationalised to consider ways of summarising and displaying the data. The students in Interview One first established three tables for three age groups but then reversed this grouping by drawing lines between individual cases of the same age. This reversal to individual cases may indicate the students were in a transition phase towards dealing with grouped data or may indicate a problem with not knowing how to deal simultaneously with a qualitative variable, age group, and two quantitative variables, fast food and weight. They did not think to calculate averages for each age group. It is debatable whether such recall would have helped in their dilemma of handling three variables.

The students in Interview Two recalled that in statistics you could find or transnumerate data into averages. With this goal in mind they split the data on gender, stratified the data into age groups and then found the average weight of each age group (Fig. 2). This separation of the data into two qualitative variables and one quantitative variable is a considerable step forward in being able to compare the averages. They did not think, however, to compare or graph the average weights. This may be because they recalled that in statistics you could find averages but did not remember the purpose for finding an average, and hence the changes in representation did not lead to more understanding. Other reasons may be that they did not define the variables, or their experience of averages has been entirely with univariate data.

The Interview Three students investigated the data systematically. They dealt with only one qualitative and one quantitative variable at a time. For the qualitative variable they found the average weight for each age group which they then graphed (Figure 3). A similar process was carried out for each activity (Figure 4). When interpreting the activity graph they were surprised at their findings as they thought the average weight for the swimming activity should be lower than the board games; "the only reason why it was high was because we didn't have enough sample ... because we only had one swimmer ... and he was quite old so he weighed quite a bit". Deliberation about sample size led them to consider that age may be affecting the average weight for each activity so they determined that the way to change the representation was to find the average weight per year of age for each activity and then compare the activities through graphical means (Fig. 5). Reference to sample size and the factoring out of the variable age demonstrates that these students are moving towards fluency in handling data, in making statistical and contextual judgements, and in transnumeration-type thinking.

All the students constructed bar graphs. The students in Interview One constructed three separate graphs (Figure 1) for each age group. Their x-axis has unordered weights since age order is retained and the data for each individual case are presented fully. The retention of information about the individual case occurred also for the students in Interview Two. Their graph (not illustrated), however, only compared two individual-named cases of the same age, on gender, weight and fast food. The students in Interview Three used their calculated averages and ratios for transnumerating into a graphical representation. The individual case has been subsumed into a particular group average or ratio. The handling of quantitative and qualitative variables poses problems for the students when they attempt to trannsnumerate the data through finding averages or constructing graphical displays. These changes in representations give another perspective on the data and can potentially lead to more understanding.

Communicating Findings

In this phase of transnumeration the students changed their statistical summaries and displays into either verbal and/or written communications. In Interview One, in response to interviewer prompts, the students first communicated by reading each graph at a local level. One student lifted the information directly from her graph: "Well the person who weighs 30-kg eats 7 fast foods per week, 24 kg's no fast foods a week, 26 kg's one fast food a week ...", whereas the other student read information about age and gender that was not explicitly in her graph: "he was 12 years old and weighed 45 kilograms and he had 5 fast food meals a week ...". The first student, with another interviewer prompt about whether the graph might tell her anything else, then reasoned between the data as well as referring to age: "It tells you that people who don't eat fast foods and considering their age at the same time ... have less weight than someone who has 7 fast foods a week but is younger." This struggle to change the representation into words involved thinking how the representation might confirm or justify her particular preconceived claim. The transnumeration culminated in their written conclusion which they considered necessary as otherwise "you wouldn't actually know what the graph is about". The conclusion was read out as: "These graphs show the amount of fast food eaten and the weight in various age groups. It proves that the more fast food you eat the more you weigh." The transnumeration was regarded as self-evident by the students at this stage.

The interviewer then transnumerated the separate graphs by putting them side-by-side and asked whether there was anything more they could say. After some time reflecting on why they drew the graphs, one student through now looking at the three graphs as one entity said:

> I just thought of something ... The older the children get it seems the more fast food they eat ... on average. Because on the 8-to-10-year-olds there's two children who eat no fast food. In the 11-to-14-year-olds there's only one person. In the 15-to18-year-olds there is no people. And the highest amount of [fast] food increases. Like um... in 8-to-10-year-olds the highest amount is seven. In the 11-to-14-year-olds the highest amount is ten and in the 15-to-18-year-olds the highest amount is 12.

The transnumeration of the graphs into this communication at a global level, or reading beyond the data, has facilitated more understanding about the data and context. The argumentation is based on a statistical statement followed by a data-based justification through comparison of maximums and minimums. A contextual evaluation of whether the statement makes sense is not articulated.

The Interview Two students communicated their findings by comparing weights of two individual cases within an age group to test their conjectured "10kg rule" for weight, which stated that they expected people to put on about 10kg each year. If the rule was followed they did not look for reasons whereas if the rule was violated they rationalised the discrepancy by finding a causal factor that explained the difference such as fast food eaten. Thus they changed the "10kg per year" representation into a causal story for the age group. In this way they were engendering understanding of the data. At no stage did they look at the subgroup as one entity. They had a global rule but gave a localised transnumeration and hence their argumentation, statistically and contextually, remained flawed.

An Interview Three student when asked to give a summary of his age and weight graph first read the information "... the 10–11[year-olds] had 30.5kgs as an average ...". When asked for further information he gave a global transnumeration: "as they grow older they obviously get heavier and weight and ... age have something to do with each other". Later on he mused: "I'm just thinking. There's like bigger leaps for some groups (he calculates the weight differences between each adjacent age group) ... like a big difference between 10–11 and 12–13. That's probably where the growth spurt is ... around 12 and 13 [years]". A shift from a global noticing of the non-linearity of the graph to a local data-based justification followed by a confirmatory contextual interpretation, is evident in this communication. The effectiveness of the transnumeration into a communication is promulgated from both a global and local perspective of the data, a comprehension of the graphs over the range of levels identified by Curcio (1987), and a form of argumentation based on statistical statements, data-based justifications, and a contextual evaluation.

Discussion

Transnumeration thinking for classifying data into relevant subgroups for comparison purpose was present. This type of thinking requires both statistical and contextual knowledge. The connecting step to then transnumerate the subgroups into defined and new entities, and the changing of quantitative variables into qualitative variables, was not well developed in the students. They found it a struggle to re-represent data into another grouping perhaps demonstrating the tension between focussing on individual cases and focussing on variables that may describe group behaviour. Bright and Friel (1998, p. 67) conjecture that "tables may play an important role as an intervening representation that can smooth the transition between representing raw and reduced data". We think, however, that the representations that re-aggregate or reclassify the raw data, which may pertain to the future analysis or to the question posed, play a role in table representations. Hancock, Kaput and Goldsmith (1992) also found that students lacked awareness of the need to represent implicit information in the data, such as gender from the given names, which they attributed to a lack of knowledge about data structures. We would conjecture that the reclassification of given data is not a feature of the taught curriculum.

The students displayed a range of responses when they constructed tables from raw data. Recognising, developing and implementing criteria for an effective classification procedure for their tables was not easy for these students, an aspect also documented in the research of Lehrer and Schauble (2000). Little research appears to have been conducted on students' construction and interpretation of statistical data tables. This study has raised questions about how students perceive tables. We believe, however, that a synergy between data, contextual knowledge, and statistical knowledge informs this transnumeration-type thinking about how to structure data.

In the structuring of data there is an intertwinement of local and global transnumeration-type thinking throughout the process of a statistical investigation. The seamless interchange between local and global thinking can be illustrated by the following hypothetical scenario. When students are presented with raw data one individual-case entry is considered at a local level. From this starting point the interchange of perspectives occurs: data are sorted on age and age becomes one global entity; data are split into local age subgroups; data are globally redefined as age-group and perceived as one entity again; each age group average is calculated locally; the averages are perceived as one global entity; each age group average is graphed and interpreted locally; the graph is perceived as one entity and interpreted globally. From such a scenario it may be possible to become aware of the current thinking of students and, if necessary, scaffold it into the next stage.

When Moritz (2000) gave bivariate and multivariate data situations to similar-aged students to graph he identified single, double, and series comparision graphs of individual cases as the possible outcomes. Students in the current study produced double comparison (Interview 2) and series comparison (Interview 1) of individual cases, and series comparison of averages and rates (Interview 3). This raises the issue of whether there is a possible developmental path for comparison of averages since the students in Interview One produced separate graphs for each age group, and the students in Interview Two calculated averages but did not graph them whereas the Interview Three students did.

Another issue pertaining to transnumerating data into graphical representations is the handling of quantitative and qualitative variables. This includes learning the types of representation for such variables, the number of variables it is possible to deal with within one representation, and factoring out or controlling variables. The students in the current study created their own graphs and ways of dealing with such variables, and were able to communicate their findings at a higher level than their representations might indicate. This suggests that students at this school level should be dealing with multivariate data, which is supported by the research of Moritz (2000). Students should be encouraged to create their own representations before being introduced to conventional ones since this will not only enable them to build up an understanding of how to manipulate and change representations but also foster creative thinking about how to produce innovative graphical representations for insights into data. This latter type of thinking was identified by Ben-Zvi and Friedlander (1997) as a possible high level of thinking that is desirable in statistics. The curriculum and teaching should encourage students to think about how to handle multivariate data..

The communication of findings was consistent with the three levels of comprehension of graphs identified by Curcio (1987), that is, reading the data, reading between the data, and reading beyond the data. For example, in Interview One a student was able, with interviewer prompts to transnumerate the graphs into a communication that shifted through each of the three levels. At the third level her language became statistical with her reasoning justified through comparing minimums and maximums within the dataset. A similar type of argumentation with a statistical statement, data-based justification to provide evidence for the statement, followed by a contextual evaluation has been described by Cobb (1999). The students in Interview Three referred to sample size but generally the students were not at the stage of generalising statistical statements and justifying statements through considering sample size. They were also not at the stage of considering such issues as whether the perceived pattern was real or random, or evaluating statistical or contextual assumptions. The communication of findings, which is a transnumeration of the statistical summaries into an evidence-based conclusion that will convince the intended audience, should be the focus of more research. In particular, research should attempt to describe the developmental levels for the type of answers that might be expected from students.

Conventional teaching at the middle-school level eschews the handling of multivariate datasets yet the ability of these students to perceive messages in such data with non-conventional graphs was quite remarkable. More research is needed on the development of transnumeration-type thinking and how transnumeration-type thinking strategies might be promoted and built up in the teaching process and curriculum. Extracting implicit messages from data requires a constant dialogue between the data, the context, and the student. Transnumeration-type thinking acts as a catalyst for this conversation.

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